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TRANSMISSION OF CONCENTRATED FORCES INTO PRISMATIC SHELLS—II

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Abstract—The paper treats the remaining two cases of an unbounded prismatic shell subjected to concentrated forces at the ridge. When the load is perpendicular to the plane of symmetry of the shell, the membrane forces remain bounded, while the moments and transverse shearing forces are singular at the load point. In contrast, the membrane forces dominate the far field. When the concentrated force is applied in the direction of the ridge, the membrane and transverse shearing forces are singular and both are of the same order at the point of loading. The far field for this case again is dominated by the membrane forces.

INTRODUCTION

THE formulation of an unbounded prismatic shell subjected to concentrated forces at the ridge and the specific loading by a force which lies in the plane of symmetry and is perpendicular to the ridge were discussed in a previous paper [12]. The present article treats the remaining two cases when the force is either perpendicular to the plane of symmetry or is applied along the ridge.

CASE II—ANTISYMMETRIC LOAD PERPENDICULAR TO RIDGE

If the concentrated force P_2 applied at the ridge is perpendicular to the plane of symmetry of the shell, as shown in Fig. 1, the deformations are antisymmetric with respect to this plane, and

$$\begin{aligned} u_x^{(1)}(x_1, y_1) &= -u_x^{(2)}(x_2, y_2) = u_x(x, y), \\ u_y^{(1)}(x_1, y_1) &= -u_y^{(2)}(x_2, y_2) = u_y(x, y), \\ w^{(1)}(x_1, y_1) &= w^{(2)}(x_2, y_2) \quad = w(x, y). \end{aligned} \tag{53}$$

Substitution of (53) into the boundary conditions (6)–(13) finds four equations satisfied identically, while the remaining relations become

$$\begin{aligned} u_x \cos\alpha + w \sin\alpha &= 0, \\ u_y &= 0, \\ M_{xx} &= 0, \\ N_{xx}\sin\alpha - V_x \cos\alpha &= -\tfrac{1}{2}P_2\delta(y). \end{aligned} \tag{54}$$

The field quantities satisfying the differential equations (3) and (5) and the boundary conditions (54) can be constructed in the same manner as for Case I. The Papkovich–

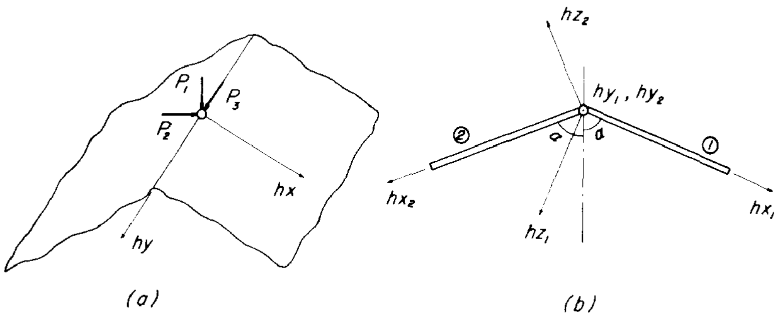

FIG. 1. Unbounded prismatic shell, loads and coordinate systems.

Neuber displacement potentials and the bending deflection constituting the desired solution are

$$\phi(x, y) = -\frac{P_2(1+\nu)}{4\pi h \sin\alpha}[\log r + C_1(x, y; a)], \tag{55}$$

$$\psi(x, y) = 0, \tag{56}$$

$$w(x, y) = \frac{P_2(3-\nu)\cos\alpha}{16\pi G h \sin^2\alpha}[2\log r + 2C_1(x, y; a) - (1-\nu)a^2 x C_0(x, y; a)], \tag{57}$$

where

$$a^2 = \frac{48\tan^2\alpha}{9-\nu^2}, \qquad (0 < \alpha < \pi/2). \tag{58}$$

The extensional displacements and membrane forces derived from (55) and (56) are

$$2Gu_x = -\frac{P_2}{4\pi h \sin\alpha}[(3-\nu)(\log r + C_1) - (1+\nu)a^2 x C_0],$$

$$2Gu_y = \frac{P_2(1+\nu)a^2}{4\pi h \sin\alpha} x S_0; \tag{59}$$

$$N_{xx} = -\frac{P_2 a^2}{4\pi h \sin\alpha}[2C_0 + (1+\nu)xC_1],$$

$$N_{xy} = -\frac{P_2 a^2}{4\pi h \sin\alpha}[(1-\nu)S_0 + (1+\nu)xS_1], \tag{60}$$

$$N_{yy} = -\frac{P_2 a^2}{4\pi h \sin\alpha}[2\nu C_0 - (1+\nu)xC_1].$$

The Airy stress function corresponding to the membrane forces (60) is

$$\chi = \frac{P_2 h}{4\pi \sin\alpha}[(1-\nu)x\log r - 2y\theta - 2C_0 - (1+\nu)xC_1]. \tag{61}$$

Finally, the moments and transverse shearing forces follow from (57) as

$$M_{xx} = \frac{P_2(1-\nu)}{2\pi(3+\nu)\cos\alpha}\left(\frac{x^2}{r^2}-a^2xC_0\right),$$

$$M_{xy} = -\frac{P_2}{2\pi(3+\nu)\cos\alpha}\left[(1-\nu)\left(\frac{xy}{r^2}-a^2xS_0\right)+(1+\nu)S_1\right], \tag{62}$$

$$M_{yy} = -\frac{P_2}{2\pi(3+\nu)\cos\alpha}\left[(1-\nu)\left(\frac{x^2}{r^2}-a^2xC_0\right)+2(1+\nu)C_1\right];$$

$$Q_x = \frac{P_2}{\pi(3+\nu)h\cos\alpha}\left(\frac{x}{r^2}-a^2C_0\right),$$

$$Q_y = \frac{P_2}{\pi(3+\nu)h\cos\alpha}\left(\frac{y}{r^2}-a^2S_0\right). \tag{63}$$

THE NEAR AND FAR FIELDS FOR CASE II

The stress resultants of the near and far fields are derived by substituting the asymptotic forms of the auxiliary functions (21)† and (25) into (60), (62) and (63). Thus the following results are obtained for the near field:

$$N^0_{xx} = \frac{1}{\nu}N^0_{yy} = -\frac{P_2a}{4h\sin\alpha}+O(r\log r),$$

$$N^0_{xy} = O(r\log r), \tag{64}$$

$$M^0_{xx} = \frac{P_2(1-\nu)}{2\pi(3+\nu)\cos\alpha}\cdot\frac{x^2}{r^2}+O(r),$$

$$M^0_{xy} = -\frac{P_2}{2\pi(3+\nu)\cos\alpha}\left[(1-\nu)\frac{xy}{r^2}+(1+\nu)\theta\right]+O(r), \tag{65}$$

$$M^0_{yy} = \frac{P_2(1+\nu)}{\pi(3+\nu)\cos\alpha}\log r+O(1);$$

$$Q^0_x = \frac{P_2}{\pi(3+\nu)h\cos\alpha}\cdot\frac{x}{r^2}+O(1),$$

$$Q^0_y = \frac{P_2}{\pi(3+\nu)h\cos\alpha}\cdot\frac{y}{r^2}+O(r\log r). \tag{66}$$

The dominant parts of the moments and transverse shearing forces near the load point are the same as the stress resultants derived from the elementary deflection function

$$w^0 = -\frac{P_2}{4\pi(3+\nu)D\cos\alpha}\left[r^2\log r-\frac{1+\nu}{1-\nu}r^2(\log r\cos 2\theta-\theta\sin 2\theta)+\frac{1}{1+\nu}r^2\right]. \tag{67}$$

† The more explicit asymptotic forms of the auxiliary functions for $r\to 0$ may be noted

$$C_0(x,y) = \pi/2a+x\log r+O(r),\qquad C_1(x,y) = -\log r+O(1),$$

$$S_0(x,y) = -y\log r+O(r),\qquad S_1(x,y) = \theta+O(r). \tag{68}$$

Apart from the multiplier $1/\cos\alpha$, w^0 given by (67) is identical to the deflection function of a semi-infinite plate loaded in bending by a concentrated force, which is applied at the edge of the plate [13].

The membrane forces in the far field are

$$\begin{aligned} N_{xx}^{\infty} &= -\frac{P_2}{4\pi h \sin\alpha}\left[(1-\nu)\frac{x}{r^2}+2(1+\nu)\frac{x^3}{r^4}\right]+O(r^{-3}),\\ N_{xy}^{\infty} &= -\frac{P_2}{4\pi h \sin\alpha}\left[(1-\nu)\frac{y}{r^2}+2(1+\nu)\frac{x^2 y}{r^4}\right]+O(r^{-3}),\\ N_{yy}^{\infty} &= -\frac{P_2}{4\pi h \sin\alpha}\left[(1+3\nu)\frac{x}{r^2}-2(1+\nu)\frac{x^3}{r^4}\right]+O(r^{-3}). \end{aligned} \tag{69}$$

The Airy stress function which gives the dominant parts of the membrane forces in the far field is

$$\chi^{\infty} = \frac{P_2 h}{4\pi \sin\alpha}[(1-\nu)x\log r - 2y\theta]. \tag{70}$$

It may be noted that the Airy stress function χ^{∞} is proportional to that of the infinitely extended plate which is loaded by an in-plane concentrated force. The bending stress resultants in the far field follow as

$$\begin{aligned} M_{xx}^{\infty} &= -\frac{P_2(1-\nu)}{\pi a^2(3+\nu)\cos\alpha}\left(3\frac{x^2}{r^4}-4\frac{x^4}{r^6}\right)+O(r^{-4}),\\ M_{xy}^{\infty} &= -\frac{2P_2}{\pi a^2(3+\nu)\cos\alpha}\left[\nu\frac{xy}{r^4}+2(1-\nu)\frac{x^3 y}{r^6}\right]+O(r^{-4}),\\ M_{yy}^{\infty} &= \frac{P_2}{\pi a^2(3+\nu)\cos\alpha}\left[(1+\nu)\frac{1}{r^2}+(1-5\nu)\frac{x^2}{r^4}-4(1-\nu)\frac{x^4}{r^6}\right]+O(r^{-4}); \end{aligned} \tag{71}$$

$$\begin{aligned} Q_{x}^{\infty} &= -\frac{2P_2}{\pi a^2(3+\nu)h\cos\alpha}\left(3\frac{x}{r^4}-4\frac{x^3}{r^6}\right)+O(r^{-5}),\\ Q_{y}^{\infty} &= -\frac{2P_2}{\pi a^2(3+\nu)h\cos\alpha}\left(\frac{y}{r^4}-4\frac{x^2 y}{r^6}\right)+O(r^{-5}). \end{aligned} \tag{72}$$

The dominant parts of the bending stress resultants given by (71) and (72) can be derived from the elementary deflection function

$$w^{\infty} = \frac{P_2(3-\nu)h^2\cos\alpha}{48\pi(1-\nu)D\sin^2\alpha}[\log r - \tfrac{1}{4}(1-\nu)\cos 2\theta]. \tag{73}$$

DISCUSSION OF RESULTS FOR CASE II

The asymptotic forms of the near field indicate the same behavior as for Case I, in that the membrane forces are bounded at the point of load application for all $\alpha < \pi/2$. According to (64)–(66), the membrane forces are of $O(1)$, the moments of $O(\log r)$ and the transverse shearing forces of $O(r^{-1})$ near the origin. The orders imply that, similarly to the previous case, the force P_2 is transferred into the shell primarily through bending action. The far-field response also resembles Case I. As seen from (69), (71) and (72), the membrane forces are of $O(r^{-1})$, moments of $O(r^{-2})$ and transverse shearing forces of $O(r^{-3})$.

The fact that the membrane forces remain bounded at the load point in Cases I and II can be explained physically. For simplicity consider $\alpha = 45°$ and a concentrated force that lies in the plane of one of the plates. Suppose that the two plates are disconnected and the force acts on them individually. The plate which the force loads in extension has displacements at the edge that are of $O(\log r)$ for $r \to 0$. In contrast, the plate which the force loads in bending suffers a deflection that is bounded and has bounded first derivatives for $r \to 0$ [13]. If the plates are loaded when they are connected, the extensional displacements cannot be larger than the transverse deflection of the plate which was loaded in bending when disconnected. Hence it can be concluded that the membrane forces must remain bounded at the load point.

The distribution of some of the stress resultants for $\alpha = 45°$ and $\nu = 1/3$ is shown in Figs. 4 and 5.

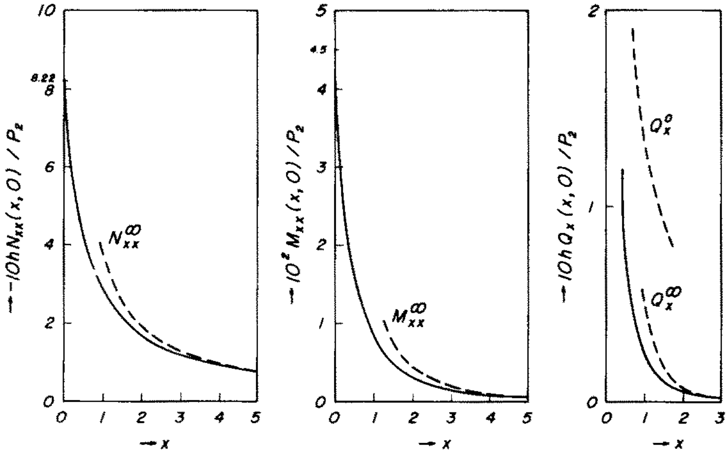

FIG. 4. Stress resultants N_{xx}, M_{xx} and Q_x along the x-axis for Case II with $\alpha = 45°$ and $\nu = \frac{1}{3}$.

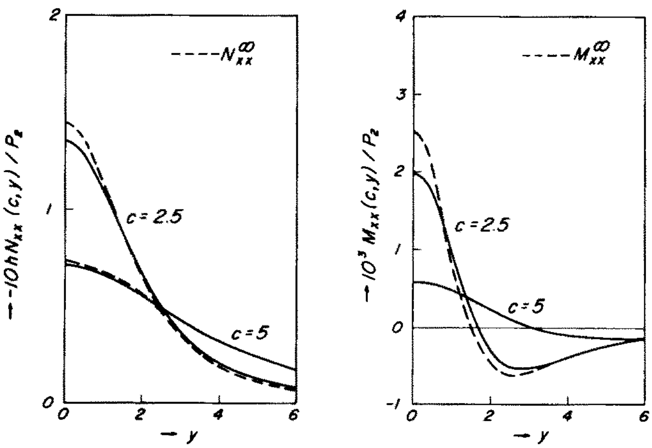

FIG. 5. Stress resultants N_{xx} and M_{xx} along a line parallel to the ridge for Case II with $\alpha = 45°$ and $\nu = \frac{1}{3}$.

CASE III—LOAD ALONG RIDGE

If the force P_3 is applied along the ridge of the prismatic shell, as illustrated in Fig. 1, the deformations are symmetric about the center plane, and equations (27) hold. Substituting (27) into the boundary conditions (6)–(13) reduces these expressions to the following:

$$\begin{aligned} u_x \sin\alpha - w\cos\alpha &= 0, \\ \frac{\partial w}{\partial x} &= 0, \\ N_{xy} &= -\tfrac{1}{2}P_3\delta(y), \\ N_{xx}\cos\alpha + V_x \sin\alpha &= 0. \end{aligned} \tag{74}$$

The Fourier exponential transform leads in the present case to the divergent integral $\int_0^\infty t^{-2}\,\mathrm{e}^{-xt}\sin yt\,\mathrm{d}t$. Proceeding as in Case I, this integral may be replaced with the harmonic function $-[y(\log r-1)+x\theta]$ which arises in considering the limits, as $\varepsilon \to 0$, of the derivatives of

$$s_{-2}(x, y; \varepsilon) = \int_\varepsilon^\infty t^{-2}\,\mathrm{e}^{-xt}\sin yt\,\mathrm{d}t, \tag{75}$$

where $x > 0$, $-\infty < y < \infty$ and $\varepsilon > 0$. The displacement potentials and the deflection function for Case III are

$$\phi(x, y) = \frac{P_3}{4\pi h}[2\theta-(1-\nu)S_1(x, y; a)], \tag{76}$$

$$\psi(x, y) = \frac{P_3}{\pi(1+\nu)}[y(\log r-1)+x\theta+\tfrac{1}{4}(1-\nu)^2 S_0(x, y; a)], \tag{77}$$

$$w(x, y) = \frac{P_3(1-\nu)\sin\alpha}{4\pi(1+\nu)Gh\cos\alpha}[\theta-S_1(x, y; a)+a^2xS_0(x, y; a)], \tag{78}$$

where

$$a^2 = 3(1-\nu^2)\cot^2\alpha, \qquad (0 < \alpha < \pi/2). \tag{79}$$

The extensional displacements and membrane forces derived from (76) and (77) are

$$\begin{aligned} 2Gu_x &= \frac{P_3}{4\pi(1+\nu)h}\left\{2(1-\nu)\theta+(1+\nu)^2\frac{xy}{r^2}+(1-\nu)[(1+\nu)a^2xS_0-2S_1]\right\}, \\ 2Gu_y &= -\frac{P_3}{4\pi(1+\nu)h}\left\{4\log r+(1+\nu)^2\frac{x^2}{r^2}+(1-\nu)[(1+\nu)a^2xC_0+(1-\nu)C_1]\right\}; \end{aligned} \tag{80}$$

$$\begin{aligned} N_{xx} &= \frac{P_3}{4\pi h}\left[(1-\nu)\frac{y}{r^2}-2(1+\nu)\frac{x^2y}{r^4}-a^2(1-\nu)(S_0+xS_1)\right], \\ N_{xy} &= \frac{P_3}{4\pi h}\left[-(3+\nu)\frac{x}{r^2}+2(1+\nu)\frac{x^3}{r^4}+(1-\nu)a^2xC_1\right], \\ N_{yy} &= \frac{P_3}{4\pi h}\left[-(3+\nu)\frac{y}{r^2}+2(1+\nu)\frac{x^2y}{r^4}-a^2(1-\nu)(S_0-xS_1)\right], \end{aligned} \tag{81}$$

where $-\pi/2 \leq \theta \leq \pi/2$. The Airy stress function corresponding to the membrane forces is

$$\chi = \frac{P_3 h}{4\pi}[2x\theta - (1-\nu)(S_0 + xS_1)]. \tag{82}$$

Finally, the bending moments and transverse shearing forces can be expressed as

$$M_{xx} = \frac{P_3(1-\nu)\cos\alpha}{8\pi\sin\alpha}\left[-(1-\nu)\frac{xy}{r^2} + (1-\nu)a^2xS_0 + (1+\nu)S_1\right],$$

$$M_{xy} = \frac{P_3(1-\nu)^2\cos\alpha}{8\pi\sin\alpha}\left[-\frac{x^2}{r^2} + a^2xC_0\right], \tag{83}$$

$$M_{yy} = \frac{P_3(1-\nu)\cos\alpha}{8\pi\sin\alpha}\left[(1-\nu)\frac{xy}{r^2} - (1-\nu)a^2xS_0 + (1+\nu)S_1\right];$$

$$Q_x = \frac{P_3(1-\nu)\cos\alpha}{4\pi h\sin\alpha}\left[-\frac{y}{r^2} + a^2S_0\right], \tag{84}$$

$$Q_y = \frac{P_3(1-\nu)\cos\alpha}{4\pi h\sin\alpha}\left[\frac{x}{r^2} - a^2C_0\right].$$

THE NEAR AND FAR FIELDS FOR CASE III

The membrane forces for $r \to 0$ are

$$N^0_{xx} = \frac{P_3}{4\pi h}\left[(1-\nu)\frac{y}{r^2} - 2(1+\nu)\frac{x^2y}{r^4}\right] + O(r\log r),$$

$$N^0_{xy} = \frac{P_3}{4\pi h}\left[-(3+\nu)\frac{x}{r^2} + 2(1+\nu)\frac{x^3}{r^4}\right] + O(r\log r), \tag{85}$$

$$N^0_{yy} = \frac{P_3}{4\pi h}\left[-(3+\nu)\frac{y}{r^2} + 2(1+\nu)\frac{x^2y}{r^4}\right] + O(r\log r).$$

The dominant parts of the membrane forces can be derived from the Airy stress function

$$\chi^0 = \frac{P_3 h}{4\pi}[(1-\nu)y\log r + 2x\theta]. \tag{86}$$

It may be noted that this is the Airy stress function for the elastic whole plane subjected to a concentrated force. This result holds for all angles $0 < \alpha \leq \pi/2$. It can be shown, however, that as $\alpha \to 0$, (81) reduce to the results for an elastic half plane which is loaded by the tangential force $\frac{1}{2}P_3$. Consequently, the membrane forces undergo a discontinuous change at $\alpha = 0$. The bending stress resultants in the near field are

$$M_{xx}^0 = \frac{P_3(1-\nu)\cos\alpha}{8\pi\sin\alpha}\left[(1+\nu)\theta-(1-\nu)\frac{xy}{r^2}\right]+O(r),$$

$$M_{xy}^0 = -\frac{P_3(1-\nu)^2\cos\alpha}{8\pi\sin\alpha}\frac{x^2}{r^2}+O(r), \tag{87}$$

$$M_{yy}^0 = \frac{P_3(1-\nu)\cos\alpha}{8\pi\sin\alpha}\left[(1+\nu)\theta+(1-\nu)\frac{xy}{r^2}\right]+O(r);$$

$$Q_x^0 = -\frac{P_3(1-\nu)\cos\alpha}{4\pi h\sin\alpha}\frac{y}{r^2}+O(r\log r),$$

$$Q_y^0 = \frac{P_3(1-\nu)\cos\alpha}{4\pi h\sin\alpha}\frac{x}{r^2}+O(1). \tag{88}$$

The dominant parts of (87) and (88) follow from the elementary deflection function

$$w^0 = -\frac{P_3(1-\nu)h^2\cos\alpha}{16\pi D\sin\alpha}[r^2\theta+\tfrac{1}{2}r^2\sin 2\theta]. \tag{89}$$

The membrane forces in the far field are

$$N_{xx}^\infty = -\frac{P_3}{\pi h}\frac{x^2y}{r^4}+O(r^{-3}),$$

$$N_{xy}^\infty = -\frac{P_3}{\pi h}\left(\frac{x}{r^2}-\frac{x^3}{r^4}\right)+O(r^{-3}), \tag{90}$$

$$N_{yy}^\infty = -\frac{P_3}{\pi h}\left(\frac{y}{r^2}-\frac{x^2y}{r^4}\right)+O(r^{-3}).$$

The dominant parts of (90) follow from the Airy stress function

$$\chi^\infty = \frac{P_3h}{2\pi}x\theta, \tag{91}$$

which is the same as the stress function for a half plane that is loaded by the tangential force $\frac{1}{2}P_3$ at the edge. Accordingly, the membrane forces in the far field are independent of the elastic constants for all $\alpha < \pi/2$. The far-field bending moments and shearing forces are

$$M_{xx}^\infty = \frac{P_3\sin\alpha}{6\pi(1+\nu)\cos\alpha}\left[\frac{xy}{r^4}-2(1-\nu)\frac{x^3y}{r^6}\right]+O(r^{-4}),$$

$$M_{xy}^\infty = \frac{P_3(1-\nu)\sin\alpha}{12\pi(1+\nu)\cos\alpha}\left(3\frac{x^2}{r^4}-4\frac{x^4}{r^6}\right)+O(r^{-4}), \tag{92}$$

$$M_{yy}^\infty = \frac{P_3\sin\alpha}{6\pi(1+\nu)\cos\alpha}\left[\nu\frac{xy}{r^4}+2(1-\nu)\frac{x^3y}{r^6}\right]+O(r^{-4});$$

$$Q_x^\infty = \frac{P_3 \sin\alpha}{6\pi(1+\nu)h\cos\alpha}\left(\frac{y}{r^4} - 4\frac{x^2 y}{r^6}\right) + O(r^{-5}),$$
$$Q_y^\infty = -\frac{P_3 \sin\alpha}{6\pi(1+\nu)h\cos\alpha}\left(3\frac{x}{r^4} - 4\frac{x^3}{r^6}\right) + O(r^{-5}). \tag{93}$$

The dominant parts of (92) and (93) can be derived from the deflection function

$$w^\infty = \frac{P_3 h^2 \sin\alpha}{24\pi(1+\nu)D\cos\alpha}(\theta + \tfrac{1}{2}\sin 2\theta). \tag{94}$$

DISCUSSION OF RESULTS FOR CASE III

The asymptotic expansions of the near field (85), (87) and (88) show that the moments are bounded, and the membrane and transverse shearing forces are $O(r^{-1})$ as $r \to 0$. These orders reveal that the behavior is different from the previous cases, and that the load which is applied along the ridge is transferred into the shell immediately through membrane action. In contrast, the nature of the far field is the same as in the former cases: the membrane forces are of $O(r^{-1})$, the moments of $O(r^{-2})$ and the transverse shearing forces of $O(r^{-3})$. A particularly interesting result is that, for all $0 < \alpha < \pi/2$, the near- and far-field membrane forces are independent of the opening angle α. In fact, the near-field membrane forces are the same as in an elastic whole plane or for the case of $\alpha = \pi/2$. Furthermore, the far-field membrane forces are identical with those existing in an elastic half plane that is subjected to a tangential force $\frac{1}{2}P_3$ at the boundary.

The distribution of N_{yy} along the ridge and along a line perpendicular to the ridge is shown in Fig. 6 for $\alpha = 45°$ and $\nu = \frac{1}{3}$.

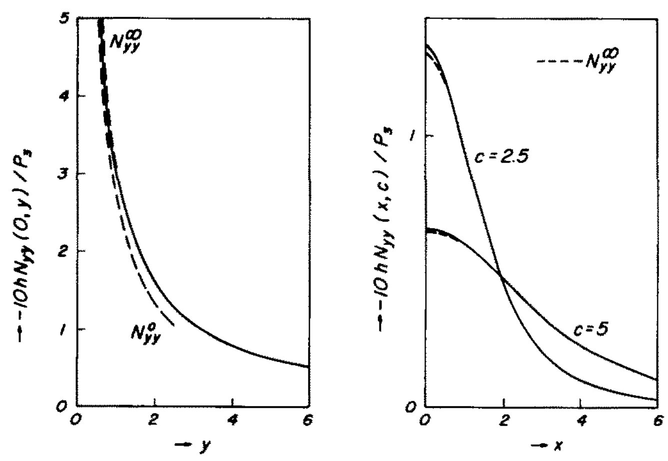

FIG. 6. Stress resultant N_{yy} along the y-axis and along a line perpendicular to the ridge for Case III with $\alpha = 45°$ and $\nu = \frac{1}{3}$.

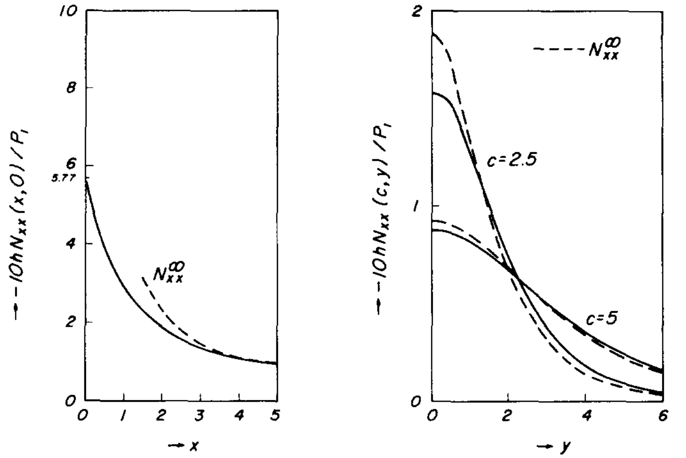

FIG. 7. Stress resultant N_{xx} along the x-axis and along a line parallel to the ridge for Case I with $\alpha = 45°$ and $\nu = \frac{1}{3}$.

ERRATUM

Some errors were discovered in Figs. 2 and 3 displaying the distributions of the stress resultants for Case I [12]. The correct results are shown in Fig. 7. It is seen that the far-field values are actually better approximations than was indicated before.

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REFERENCES

[12] J. DUNDURS and M. G. SAMUCHIN, Transmission of concentrated forces into prismatic shells—I. *Int. J. Solids Struct.* 7, 1627 (1971).
[13] A. NÁDAI, *Elastische Platten*. Springer (1925).

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Абстракт—Работа обсуждает остальные два случаи неограниченных призматических оболочек, подверженных действию концентрических усилий на грани. Когда нагрузка перпендикулярна к плоскости симметрии оболочки, тогда мембранные усилия конечны, но моменты и поперечные сразывающие силы сингулярны в точке приложения нагрузки. В противоположности мембранные усилия преобладают на большом расстоянии от места приложения нагрузки. Для случая концентрического усилия, приложенного по направлению грани, мембранные и поперечные сразывающие усилия ангулярны. Они также этого самого порядка, в точке приложения нагрузки. Но, для этого случая мембранные усилия преобладают в областях отдаленных от точки нагружения.